On the Order of the Group bP_{4k}

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Summary of some of the things in Kervaire Milnor. We want to cover the content in [KM63] in particular section §7 and [KM13].

1 Survey of [KM63]

First all manifolds are compact, smooth and oriented. All maps are orientation preserving. The aim of this paper is to prove that the set of h-cobordism classes of homotopy spheres of dimension n, Θ_n, (closed manifolds with the homotopy type of an n sphere) is finite. One reason to do this is because for $n \neq 4$ it is known that this set is in bijection with the set of differentiable structures on the n sphere. Thus we are bounding how many exotic spheres there are. This is done using group theory.

§2 is dedicated to showing that Θ_n is a group under the operation of (smooth) connected sum. The key result in this section that we need to highlight is the following criteria for something to be h-cobordant to the standard sphere. By the h-cobordism theorem this is the same as being diffeomorphic to the standard sphere, or in this context to being in the equivalance class of the identity in Θ_n:

Lemma. *A simply connected manifold is h-cobordant to the sphere S^n iff it bounds a contractable manifold.*

One way to understand this is that a contractable manifold with a simply connected boundary (by the h-cobordism theorem) is diffeomorphic to the standard disc (see my notes on smooth structures on the sphere and h-cobordism). Anyway KM show that the operation is well defined on h-cobordism classes, and defines a group operation (checking inverses, associativity and identity). It is also clear that the operation is abelian.

§3 is a discussion of the concept of stabily parallelizability, defining it and giving some necissary and / or sufficient conditions for things to be s-parallelizable. §4 defines the subgroup $bP_{n+1} \subseteq \Theta_n$ of homotopy spheres that bound parallelizable manifolds (in this case it is equivalent to be s-parallelizable or just parallelizable). They then use the Pontryagin-Thom construction to define a group homomorphism

$$p : \Theta_n \to \pi_n^s$$

into the stable n stem, such that its kernel is bP_{n+1}, hence we have that $\Theta/bP_{n+1} \subseteq \pi_n^s$ which is finite by Serre. Recal the the Pontryagin-Thom construction defines a bijection between

$$\big(\text{Maps } M \to S^n/\text{smth homotopy}\big) \to \text{Manifolds/framed cobordism}$$

$$f \mapsto f^{-1}(p), \quad p \text{ a regular value}$$

The boundary of a parallelizable manifold can be immbeded in S^{n+k} and then we consider the map $S^{n+k} \to S^k$ given by the Pontryagin-Thom construction on S^{n+k} with the framing coming from the trivialisation of the normal bundle of the imbedded boundary. This is an element of the required stable stem.

What remains is to show that bP_{n+1} is finite, this will imply that Θ_n is finite, since its quotient by a finite thing is finite. The argument is now indexed by $n+1$ and is an argument mod 4. For odd $n+1$ then $bP_{n+1} = 0$. For $n+1 \equiv 2 \ (mod\ 4)$ then it is either trivial or $\mathbb{Z}/2\mathbb{Z}$. So the most interesting case is $n+1 \equiv 0$ mod 4.

2 A Sketch of $n \equiv 0, 1, 2$

Just a sketch, insofar as it is relevant to our understanding of the modulo 3 case, and summary.

2.1 Odd values of $n+1$

This is the meat of the paper and is dealt with through §5 for the $n \equiv 0$ case and §6 for $n \equiv 2$ case. It is still not clear to me why these two cases are seperated, and in particular why n=2 we have to deal with orientation issued but not in the n=0 case? The idea in these sections is to apply the lemma above and show that the homotopy spheres of dimension n bound a contractable manifold (and are therefore trivial).

The setup is now that we have a dimension n homotopy sphere, where $n \equiv 0, 2$ and we assume that it bounds an s-parrallelizable manifold of dimension $n+1$. That is we consider M an s-parallelizable $n+1$ manifold with boundary a homotopy sphere. We now "perform surgery" to M to construct an M' which has the same boundary but is contractable. This implies that the boundary, our homotopy sphere is S^n.

The process is as follows, first it is sufficient to construct a space whose homotopy groups are all zero becuase by Whiteheads theorem this implies that the space is homotopic to the disc, that is in particular contractable. We need something that has the same boundary and whose homotopy groups are smaller; Milnors construction is for $n = p+q+1$ to consider an imbedding

$$\varphi : S^p \times D^{q+1} \to M$$

and then construct $\chi(M, \varphi)$ as

$$\big(M \setminus \varphi(S^p \times 0) \sqcup D^{p+1} \times S^q\big)/\big(\varphi(u, tv) \sim (tu, v), \ \ u \in S^p, v \in S^q, t \in (0,1]\big)$$

Remark. Notice that the disjoint union is with $D^{p+1} \times S^q$, in particular the boundary of the first terem is the thing which we have imbedded, S^p, while the other thing we have immbedded D^{q+1} has the boundardy S^q. Picture here.

Note here that infact the disjoint union of M and $\chi(M, \varphi)$ boundas another manifold, the so called surgeons suitcase. In particular these two manifolds are co-bordant. They are not diffeomorphic as the cobordism is not necissarily simply connected (so cant apply h-cobordism).

Lemma. *The boundary of* $\chi(M,\varphi)$ *is equal to the boundary of* M *and moreover for* $p<q$ *or* $p \leq n/2-1$ *we have that*

$$\pi_i(\chi(M,\varphi)) = \begin{cases} \pi_i(M), & i<p \\ \pi_i(M)/\langle \varphi|_{S^p\times 0}\rangle, & i=p \end{cases}$$

Thus spherical modification *below the middle dimension* is sufficient to kill homotopy groups represented by embeddings. When M is s-parallelizable and $p < n/2$ all homotopy elements are represented by some such embedding.

So now we have a manifold of dimension $n+1$ that is s-parallelizable and whose boundary is a homotopy sphere. We know that all the elements of its homotopy groups for π_i are representable by embeddings of this form for $i < n/2+1/2$. Finally we know that for $i < n/2-1/2$ we can kill the homotopy group elements represented by these embeddings. Thus we may assume that we have an $n+1$ dimensional manifold that is $n/2-1/2$ connected, that is s-parallelizable and whose boundary is a homotopy sphere. Because we have that $n \equiv 0,2 \bmod 4$, we see that if $n = 4k, 4k+2$ then the space is $2k-1/2$ or $2k+1/2$ connected respectively. In particular we see that if we are in the 2 mod 4 case we have that we are $2k$ connected while in the 0 mod 4 case we are only $2k-1$ connected at this stage. We see that the difference between these two cases is the connectedness in *the middle dimension*. There is another dimensional restriction, in that we only know a priori that for $n \geq 2p$ the modified manifold is still s-parallelizable. So there are clearly some subtelties to deal with, with regards to s-parallelizability and the middle dimension.

The final part of the argument is to apply Hurewicz theorem to our highly connected space and instead of attacking the homotopy groups in the middle dimension we consdier the cohomology groups in the middle dimension. Notice by Poincare duality we also know that the *higher* cohomology groups are also already zero and so all that remains is the middle dimensions. Indeed cohomological techniques (exact sequences) are used to show that spherical modification can kill the middle dimension cohomology which then proves that all the homotopy groups are zero.

2.2 Congruent to 2

This is §8, and I didnt really go through any details. The idea is to define a cohomology operation and then use it to prove a sufficient condition for triviality. The next thing is to show that non-trivial things are related to the Arf invariant which lands in $\mathbb{Z}/2\mathbb{Z}$. Something like that.

3 The Order of bP_{4k}

This is dealt with in §7 in [KM63] however the details of this calculation are largely outsourced to [KM13]. In this setting we are considering a $4k$ dimensional manifold, whose boundary is a homotopy sphere. Because we are in a dimension that is a multiple of 4 we can use the signature to our advantage.

Lemma (Lem 7.3)**.** *Let M be as above, then its homotopy groups can be killed by a series of framed spherical modifications iff the signature is zero.*

That is the kernel of the signature is exactly the manifolds that can be surged to the disk. Note that the forward direction here is simply that surgery creates a manifold co-bordant to the original manifold and since the signature is a co-bordism invariant it doesnt change the signature. The converse relies on the analysis of §6 to show when framed spherical modifications are available. I think understanding this lemma more deeply could be worthwhile.Lance thm 5.2, how does that fit with what I have said here?

3.1 Some Black Boxes

An almost parallelizable manifold is a manifold with a designated point M^n, x_0 such that $M - x_0$ is paralllelizable. Given an immbedding into $\mathbb{R}^k$, for $k > 2n+1$,this is equivalent to the normal bundle of $M - x_0$ being trivial.

We will take the following facts about almost parallelizable manifolds and the J homomorphism for granted:

1. $\pi_{4n-1}(SO_m) \cong \mathbb{Z}$, by Bott periodicity, for $m > 4n$.

2. The J homomorphism is defined on $\pi_{4n-1}(SO_m) \to \pi_{4n-1+m}(SO_m)$, and has a finite cyclic image, therefore for some $j_n \in \mathbb{Z}$ we can (by above) identify it with a map

$$J : \mathbb{Z} \to \mathbb{Z}/j_n\mathbb{Z}$$

3. For a connected almost parallelizable $4n$ manifold the top cohomology is

$$H^{4n}(M; \pi_{4n-1}(SO_m)) \cong \pi_{4n-1}(SO_m) \cong \mathbb{Z}$$

   Proof. The coefficient group is still just $\mathbb{Z}$ by above and then this follows from Poincare duality, as connectedness implies that there is one path component and so the zeroeth homology is just the free abelian group on the path components which is just $\mathbb{Z}$.

4. If we embed M^{4n} into $\mathbb{R}^{m+n}$ for large m, and f is a section of the normal bundle of $M - x_0$ then the obstruction to extending f to all of M is a class

$$\mathfrak{o} \in H^{4n}(M; \pi_{4n-1}(SO_m))$$

5. For a $4n$ almost parallelizable manifold all $p_i[M] = 0$ for $0 < i < n$, that is only the top classes can be non-zero.

   Proof. [Kos93, IX. (8.2)] shows that every an almost trivial bundle is always the pullback along a degree one map of a bundle over the sphere (collapse map). So in particular the tangent bundle of an almost parallelizable manifold, which is by definition almost trivial, is such a pullback. Thus the characteristic classes, as they are natural in pullbacks, are characteristic classes of some bundle over the sphere, which must be zero in degrees below the top dimension (as the cohomology groups *of the sphere* are zero).

3.2 Almost Parallelizable Manifolds

In [KM13] follow a proof of Rohlin and simply ask which steps generalise. Denote by

$$a_n = \begin{cases} 2, & n \text{ odd} \\ 1, & \text{else} \end{cases}$$

then [KM13] show that

Theorem. *The Pontryagin numbers define a surjective map*

$$\textit{almost parrallelizabl } 4n \textit{ manifolds} \to j_n a_n (2n-1)!\mathbb{Z}$$

In particular the image is a subgroup of $\mathbb{Z}$ *.*

Step 1. First we need to show that the kernel of the J homomorphism is precisely the homotopy classes which correspond to obstructions to parallalizing an almost parallelizable manifold, this makes sense applying the isomorphism of (3) above. To make this precise we have:

Lemma. *Given a section of the normal bundle of* $M^{4n} - x_0 \subseteq \mathbb{R}^{m+n}$*,* f*, then*

$$J(\mathfrak{o}(f)) = 0$$

Where we identify $\mathfrak{o} \in H^{4n}(M; \pi_{4n-1}(SO_m))$ *with its image in either* $\pi_{4n-1}(SO_m)$ *or* $\mathbb{Z}$ *.*

They prove something much stronger, but we dont need it.

Proof. Uses the Pontryagin-Thom construction and the definition of the J-homomorphism given by twisting such framed cobordism classes.

Step 2. Next we relate the Pontryagin numbers of almost parallelizable manfolds to their obstruction classes. In particular

Lemma. *If we denote* ν *the normal bundle of* $M^{4n} \subseteq \mathbb{R}^{m+n}$ *and we take a section of this normal bundle restricted to* $M - x_o$ *call it* f *then*

$$p_n(\nu) = \pm a_n (2n-1)!\, \mathfrak{o}(f)$$

Again they prove a much stronger statement.

Proof. Relate the SOm bundle to Um bundles. Um obstructions are chern classes/pontryagin classes up to a sign. Pullback the Pontryagin classes via the maps relating SOm bundles to Um bundles and use natrality to relate them. Computations of Bott give explicitly what these pullbacks are, they are multiplication by the required numbers.

Step 3. Whitney sum formula relates the normal and tangent bundles Pontryagin classes as

$$2 = 2p(\nu \oplus \tau) = 2p(\nu)p(\tau)$$

because their sum is the trival bundle, moreover because the top cohomology is known to be $\mathbb{Z}$ we can in fact cancel the two away to see that

$$p(\tau) = p(\nu)^{-1}$$

Because all lower classes are zero we get that in fact, $1 + p_n(\tau) = (1 + p_n(\nu))^{-1}$, which we can see

$$(1+\alpha)(1-\alpha) = 1 - \alpha^2 = 1$$

because α^2 is in too high a cohomology class. Thus $p_n(\tau) = -p_n(\nu)$, or

$$p_n(\tau) = \pm a_n(2n-1)!\ \mathfrak{o}(f)$$

From the first lemma $\mathfrak{o}$ is in the kernel of J and hence is divisible by j_n. In particular there is some element $\alpha \in H^{4n}(M; \pi_{4n-1}(SO_m)) \cong \mathbb{Z}$ such that $\mathfrak{o} = j_n\alpha$, hence

$$p_n(\tau) = \pm a_n(2n-1)!\ j_n\alpha$$

This shows that the image of the Pontryagin numbers is certainly in the subgroup we claimed. To see surjectivity it is sufficient to find an element mapping to the generator of this subgroup, as Pontryagin numbers are additive, namely mapping to $j_n a_n(2n-1)!$. By definition we have the following ses

$$0 \to j_k\mathbb{Z} \to \pi_{4k-1}(SO) \cong \mathbb{Z} \to \mathbb{Z}/j_k\mathbb{Z} \to 0$$

and because j_k is in the kernel of J we know by lemma 1 that there is some required manifold corresponding to it.

Remark. [Kos93, Thm. 8.7] gives some more explicit construction of such a manifold.

Remark. It is an immediate corrolory by applying the Hurzebruch signature theorem and the fact that lower Pontryagin classes are zero that the signature defines a surjective map

$$\text{almost parrallelizable } 4n \text{ manifolds} \to 2^{2n-1}(2^{2n-1}-1)B_n j_n a_n/n\mathbb{Z}$$

Remark. This number can be shown by other means to be an integer, thus proving that the order of $J\pi_{4n-1}(SO)$ is a multiple of the denominator of $B_n a_n/4n$.

Remark. In particular this manifold has a non-zero signature, note that it is by assumption connected and without boundary (I think). Given this $4k$ manifold we can simply remove the interior of a $4k$ disk around x_0 which gives us a parallelizable manifold whose signature is non-zero (see below) and whose boundary is in fact the standard sphere. Notice that in fact it is the standard sphere that is bounding a manifold that cannot be made to be contractible via spherical modifications, so in particular the fact that spherical modifications cant be used is not sufficient to show that the element is non-trivial in the group Θ_n.

3.3 Back to bP_{4k}

Lemma. *The signature defines a map from all $4k$ manifolds that are which are s-parallelizable and bound the standard $4k-1$ sphere to the integers. The image of this map is $2^{2m-1}(2^{2m-1}-1)B_m j_m a_m/m\mathbb{Z}$.*

We will denote the integer $\sigma_m = 2^{2m-1}(2^{2m-1}-1)B_m j_m a_m/m > 0$ the generator of this subgroup.

Proof. By the analysis in the previous section it is sufficient to see that the signature does not change when we remove the interior of a disc from a manifold. This is clear because given the almost parallelizable manifolds above we can simply remove the interior of a disc around the base point and therefore obtain a (s) parallelizable manifold with the required signature.

We recall that the signature of a manifold is just the signature of the pairing

$$H^{2n}(M;\mathbb{Q}) \otimes H^{2n}(M;\mathbb{Q}) \to \mathbb{Q}$$

$$(a,b) \mapsto (a \smile b)[M]$$

If we consider the inclusion $D^{4n} \hookrightarrow M^{4n}$ then by the LES in cohomology we get that

$$\cdots \to H^{i-1}(D^{4m}) \to H^i(M, D^{4n}) \to H^i(M) \to H^i(D^{4n}) \to H^{i+1}(M, D^{4n}) \to \cdots$$

because we are in the case of $4n$ for $n > 1$ the middle dimension of $2n$ is such that $H^{2n\pm 1}(D^{4n}) = 0$ and thus we get that $H^{2n}(M, D^{4n}) \cong H^{2n}(M)$. Next apply excision to the pair $(M - D^n, \emptyset) \hookrightarrow (M, D^{4n})$ to see that

$$M^{2n}(M, D^{4n}) \cong H^{2n}(M - D^{4n})$$

Thus we see that $H^{2n}(M) \cong H^{2n}(M - D^{4n})$. Both of these isomorphisms clearly hold in dimension $4n$ as well and so we have the result. We now pray that the isomorphisms given by excision and the LES are natural (commute) with respect to cup products and the result follows.

Lemma. *Let $(M, bM) = (M_1, bM_1)\sharp(M_2, bM_2)$ be the connected sum along the boundary of two s-parallelizable manifolds bounded by homotopy spheres. Then $\sigma(M) = \sigma(M_1) + \sigma(M_2)$.*

Proof. Discussed here. [DFN90, Thm. 27.5] proves the simplest case, that is however not sufficient here, that is additivity when we glue the full boundary. [Wal69] states that glueing along a submanifold of the boundary that does not itself have a boundary also works, using the same proof. The proof relies on some decompositions of the homology into a direct sum of peices that are related to the peices of the gluing and the gluing maps.

In the KM case however we have that $M \sim M_1 \vee M_2$ that is homotopy type of the gluing of M_1, M_2 at a single point (they are glued along a disk which is contractible). The signature is also a homotopy invariant (according to [MS16, Cor 19.6]). It is then clear from Mayer-Vietoris that

$$H^{4k}(M_1 \vee M_2) \cong H^{4k}(M_1) \oplus H^{4k}(M_2)$$

which we again pray is natural in the cup product. This also clearly holds in dim $2k$ as well.

Lemma. *Let $[\Sigma_1], [\Sigma_2] \in bP_{4k}$ that bound s-parallizable manifolds M_1, M_2 respectively. Then $[\Sigma_1] = [\Sigma_2]$ iff $\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m}$.*

Proof. Let Σ_1, Σ_2 and M_1, M_2 be as in the statement.

Forward: Consider a h-cobordism W between $-\Sigma_1\sharp\Sigma_2$ and the standard sphere (which exists because $\Sigma_1 \cong \Sigma_2$ by hypothesis). Glue W onto $(-M_1, -bM_1)\sharp(M_2, bM_2)$ along the common boundary given by $-\Sigma_1\sharp\Sigma_2$, that is the connected sum that also connected sums the boundary. This gives a manifold bounded by the sphere S^{4m-1} that we denote M. M is s-parallelizable (we are in the case that s-parallelizable iff parallelizable, and it is intuitively clear that the connected sum of parallelizable manifolds should be parallelizable) and so we know by the previous lemma that its signature lands in $\sigma_m\mathbb{Z}$ or equivalently that

$$\sigma(M) \equiv 0, \pmod{\sigma_m}$$

But by the additivity of the signature we have that

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2)$$

and so we get that

$$\sigma(M_1) \equiv \sigma(M_2), \pmod{\sigma_m}$$

Converse: Assume that $\sigma(M_1) \equiv \sigma(M_2), \pmod{\sigma_m}$, in particular there is some s-parallelizable $4k$ manfiold M_0 bounded by the sphere such that

$$\sigma(M_1) = \sigma(M_2) + \sigma(M_0)$$

Again we take the connected sum along the boundary

$$(M, bM) := (-M_1, -bM_1)\sharp(M_2, bM_2)\sharp(M_0, bM_0)$$

which has boundary given by the connected sum of all the boundary components

$$-\Sigma_1 \sharp \Sigma_2 \sharp S^{4k-1} \cong -\Sigma_1 \sharp \Sigma_2$$

as the standard sphere is the unit in the group of homotopy spheres. Taking the signature of this connected sum we get that

$$\sigma(M) = \sigma(M_0) - \sigma(M_1) + \sigma(M_2) = 0$$

hence by the lemma above we may use surgery to kill the homotopy groups of M and therefore construct a contractable manifold that is bounded by $-\Sigma_1 \sharp \Sigma_2$ which is therefore h-cobordant to the standard sphere. Thus by the group operations we have that

$$-\Sigma_1 \sharp \Sigma_2 \cong S^{4k-1} \implies \Sigma_2 \cong \Sigma_1 S^{4k-1} \implies \Sigma_1 \cong \Sigma_2$$

That is if we mod out by the pathological spaces which bound the standard sphere then the signature is a complete invariant for homotopy spheres being diffeomorphic. This amounts to showing that the map

$$bP_{4k} \to 4k \text{ s- parallelizable manifolds bounded by a homotopy sphere} \to \mathbb{Z} \to \mathbb{Z}/\sigma_m\mathbb{Z}$$

$$\Sigma \mapsto M \text{ such that } \partial M = \Sigma \mapsto \sigma(M) \mapsto [\sigma(M)]$$

is a well defined group , where s-parallelizable spaces bounded by a homotopy sphere are given the operation connected sum along the boundardy. Moreover this map is *injective*, as if the signature is zero we know that by Lem 7.3 the manifold can be made to be contractible and hence the space in bP_{4k} bounds a contractible manifold and is therefore the standard sphere. Hence we get that bP_{4k} for $k > 1$ is isomorphic to a subgroup of $\mathbb{Z}/\sigma_m\mathbb{Z}$, which is itself finite cyclic and therefore we can conclude that bP_{4k} is finite, cyclic and that its order divides σ_m.

Remark. [KM63] claim that an integer occurs as a signature in this way iff it is congruent to 0 modulo 8, they give no reference saying it will appear in part 2. This would imply that in particular the order of bP_{4k} is exactly $\sigma_k/8$.Go find a proof of this.

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